Solution to Mid-term Exam, MMAT5520

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1.(6 marks) Solve the initial value problem

$$x\frac{dy}{dx} + 3y - 8x^2 = 0,\ x > 0;\ y(1) = 5.$$

Soution: Multiplying x^2 on both sides of the equation

$$\begin{aligned} x^3\frac{dy}{dx} + 3x^2 y &= 8x^4, \\ \frac{d}{dx}(x^3 y) &= 8x^4, \\ x^3 y &= \int 8x^4 dx, \\ x^3 y &= \frac{8}{5}x^5 + C, \\ y &= \frac{8}{5}x^2 + Cx^{-3}. \end{aligned}$$

Since $y(1) = 5, C = \frac{17}{5}$. Thus

$$y = \frac{8}{5}x^2 + \frac{17}{5}x^{-3}.$$

2.(6 marks) Solve

$$\frac{dy}{dx} = \frac{y + \sqrt{xy}}{x},\ x > 0.$$

Soution:
Rewriting the equation as

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{\frac{y}{x}}.$$

Let $u = \frac{y}{x}$, we have

$$\begin{aligned} u + x\frac{du}{dx} &= u + \sqrt{u}, \\ x\frac{du}{dx} &= \sqrt{u}, \\ \frac{du}{\sqrt{u}} &= \frac{dx}{x}, \\ \int \frac{du}{\sqrt{u}} &= \int x^{-1} dx, \\ 2\sqrt{u} &= \ln|x| + C, \\ 2\sqrt{\frac{y}{x}} &= \ln|x| + C, \\ y &= \frac{1}{4}x(\ln|x| + C)^2 \quad \text{or} \quad y = 0. \end{aligned}$$

3.(6 marks) Show that the equation

$$(4xy + 2y^2)dx + (x^2 + 3xy)dy = 0$$

has an integrating factor of the form $\mu(x, y) = y^k$ and solve the equation.

Soution: Multiplying the equation by $\mu(x, y) = y^k$ gives

$$(4xy^{k+1} + 2y^{k+2})dx + (x^2y^k + 3xy^{k+1})dy = 0.$$

Now

$$\begin{aligned} \frac{\partial}{\partial y}(4xy^{k+1} + 2y^{k+2}) &= 4(k + 1)xy^k + 2(k + 2)y^{k+1} \\ \frac{\partial}{\partial x}(x^2y^k + 3xy^{k+1}) &= 2xy^k + 3y^{k+1} \end{aligned}$$

Let $k = -\frac{1}{2}$, then

$$\frac{\partial}{\partial y}(4xy^{\frac{1}{2}} + 2y^{\frac{3}{2}}) = 2xy^{-\frac{1}{2}} + 3y^{\frac{1}{2}} = \frac{\partial}{\partial x}(x^2y^{-\frac{1}{2}} + 3xy^{\frac{1}{2}}).$$

The equation is exact.

Set

$$F(x, y) = \int (4xy^{\frac{1}{2}} + 2y^{\frac{3}{2}})dx = 2x^2y^{\frac{1}{2}} + 2xy^{\frac{3}{2}} + g(y).$$

We want

$$\begin{aligned} \frac{\partial F(x, y)}{\partial y} &= x^2y^{-\frac{1}{2}} + 3xy^{\frac{1}{2}}, \\ x^2y^{-\frac{1}{2}} + 3xy^{\frac{1}{2}} + g'(y) &= x^2y^{-\frac{1}{2}} + 3xy^{\frac{1}{2}}, \\ g'(y) &= 0. \end{aligned}$$

Therefore we may choose $g(y) = 0$ and the solution is

$$2x^2y^{\frac{1}{2}} + 2xy^{\frac{3}{2}} = C.$$

4. (6 marks) Let $A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 0 \\ -1 & -2 & 2 \end{pmatrix}$. Find A^{-1} by

(a) using elementary row operations.

(b) finding the adjoint of A.

Soution: (a)

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 3 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 5 & 3 & 1 & 0 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3\to R_3+R_1]{R_2\to R_2-2R_1}\left(\begin{array}{ccc|ccc}1&2&0&0&1&0\\0&1&3&1&-2&0\\0&0&2&0&1&1\end{array}\right)\xrightarrow{R_3\to\frac{1}{2}R_3}\left(\begin{array}{ccc|ccc}1&2&0&0&1&0\\0&1&3&1&-2&0\\0&0&1&0&\frac{1}{2}&\frac{1}{2}\end{array}\right)$$

$$\xrightarrow{R_2\to R_2-3R_3}\left(\begin{array}{ccc|ccc}1&2&0&0&1&0\\0&1&0&1&-\frac{7}{2}&-\frac{3}{2}\\0&0&1&0&\frac{1}{2}&\frac{1}{2}\end{array}\right)\xrightarrow{R_1\to R_1-2R_2}\left(\begin{array}{ccc|ccc}1&0&0&-2&8&3\\0&1&0&1&-\frac{7}{2}&-\frac{3}{2}\\0&0&1&0&\frac{1}{2}&\frac{1}{2}\end{array}\right).$$

Therefore

$$A^{-1}=\begin{pmatrix}-2&8&3\\1&-\frac{7}{2}&-\frac{3}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}.$$

(b)

$$\det A=\begin{vmatrix}2&5&3\\1&2&0\\-1&-2&2\end{vmatrix}=\begin{vmatrix}0&1&3\\1&2&0\\0&0&2\end{vmatrix}=-2,$$

and

$$adj\,A=\begin{pmatrix}\begin{vmatrix}2&0\\-2&2\end{vmatrix}&-\begin{vmatrix}5&3\\-2&2\end{vmatrix}&\begin{vmatrix}5&3\\2&0\end{vmatrix}\\-\begin{vmatrix}1&0\\-1&2\end{vmatrix}&\begin{vmatrix}2&3\\-1&2\end{vmatrix}&-\begin{vmatrix}2&3\\1&0\end{vmatrix}\\\begin{vmatrix}1&2\\-1&-2\end{vmatrix}&-\begin{vmatrix}2&5\\-1&-2\end{vmatrix}&\begin{vmatrix}2&5\\1&2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}4&-16&-6\\-2&7&3\\0&-1&-1\end{pmatrix}.$$

Therefore

$$A^{-1}=\frac{adj\,A}{\det A}=-\frac{1}{2}\begin{pmatrix}4&-16&-6\\-2&7&3\\0&-1&-1\end{pmatrix}=\begin{pmatrix}-2&8&3\\1&-\frac{7}{2}&-\frac{3}{2}\\0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}.$$

5. (6 marks) Find the equation of the circle of the form $x^2+y^2+Dx+Ey+F=0$ which passes through $(2,-1)$, $(0,-3)$ and $(-2,3)$ by writing down a suitable determinant.

Soution: The equation of required circle is

$$\begin{vmatrix}1&x&y&x^2+y^2\\1&2&-1&2^2+(-1)^2\\1&0&-3&0^2+(-3)^2\\1&-2&3&(-2)^2+3^2\end{vmatrix}=0,$$

$$\begin{vmatrix}1&x&y&x^2+y^2\\1&2&-1&5\\1&0&-3&9\\1&-2&3&13\end{vmatrix}=0,$$

$$\begin{vmatrix}1&x&y&x^2+y^2\\1&2&-1&5\\0&-2&-2&4\\0&-4&4&8\end{vmatrix}=0,$$

$$\begin{vmatrix}1&x&y&x^2+y^2\\1&2&-1&5\\0&1&1&-2\\0&1&-1&-2\end{vmatrix}=0,$$

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & x & x^2+y^2 \\ 1 & 2 & 5 \\ 0 & 1 & -2 \end{vmatrix} = 0,$$

$$x^2 + y^2 + 2x - 9 = 0.$$

6. (6 marks)Let $\mathbf{M}$ be a 4×4 matrix with $\det(\mathbf{M}) = m \neq 0$. Write $\mathbf{M} = [\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4]$, where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are the column vectors of $\mathbf{M}$. Find the determinant of the following matrices in terms of m.

(a) $\mathbf{A} = [\mathbf{x}_4,\ 4\mathbf{x}_2 - \mathbf{x}_3,\ \mathbf{x}_3,\ \mathbf{x}_1]$

(b) $\mathbf{B} = \mathbf{M}^T\mathbf{M}^2$

(c) $\mathbf{C} = 3\mathbf{M}^{-1}$

Soution: (a)

$$\begin{aligned} \det \mathbf{A} &= |\mathbf{x}_4,\ 4\mathbf{x}_2 - \mathbf{x}_3,\ \mathbf{x}_3,\ \mathbf{x}_1| \\ &= -|\mathbf{x}_1,\ 4\mathbf{x}_2 - \mathbf{x}_3,\ \mathbf{x}_3,\ \mathbf{x}_4| \\ &= -|\mathbf{x}_1,\ 4\mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4| \\ &= -4|\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4| \\ &= -4m. \end{aligned}$$

(b) $\det \mathbf{B} = \det(\mathbf{M}^T\mathbf{M}^2) = \det(\mathbf{M}^T) \cdot \det \mathbf{M} \cdot \det \mathbf{M} = \det \mathbf{M} \cdot \det \mathbf{M} \cdot \det \mathbf{M} = m^3.$

(c) $\det \mathbf{C} = \det(3\mathbf{I}_4\mathbf{M}^{-1}) = \det(3\mathbf{I}_4)(\det \mathbf{M})^{-1} = 3^4 m^{-1} = 81m^{-1}.$

7. (6 marks) Let $\mathbf{A} = \begin{pmatrix} 1 & -2 & -2 & 1 & 3 \\ 3 & -6 & -1 & 2 & 4 \\ 1 & -2 & 5 & 8 & -4 \\ -2 & 4 & -3 & 1 & 1 \end{pmatrix}$.

(a) Find a basis for the row space of $\mathbf{A}$.

(b) Find a basis for the column space of $\mathbf{A}$.

(c) Find a basis for the null space of $\mathbf{A}$.

Soution:

$$\begin{pmatrix} 1 & -2 & -2 & 1 & 3 \\ 3 & -6 & -1 & 2 & 4 \\ 1 & -2 & 5 & 8 & -4 \\ -2 & 4 & -3 & 1 & 1 \end{pmatrix} \xrightarrow[R_4 \to R_4 + 2R_1]{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1}} \begin{pmatrix} 1 & -2 & -2 & 1 & 3 \\ 0 & 0 & 5 & -1 & -5 \\ 0 & 0 & 7 & 7 & -7 \\ 0 & 0 & -7 & 3 & 7 \end{pmatrix}$$

$$\xrightarrow{R_3\to\frac{1}{7}R_3}\begin{pmatrix}1&-2&-2&1&3\\0&0&5&-1&-5\\0&0&1&1&-1\\0&0&-7&3&7\end{pmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{pmatrix}1&-2&-2&1&3\\0&0&1&1&-1\\0&0&5&-1&-5\\0&0&-7&3&7\end{pmatrix}$$

$$\xrightarrow[R_4\to R_4+7R_2]{R_3\to R_3-5R_2}\begin{pmatrix}1&-2&-2&1&3\\0&0&1&1&-1\\0&0&0&-6&0\\0&0&0&10&0\end{pmatrix}\xrightarrow{R_3\to-\frac{1}{6}R_3}\begin{pmatrix}1&-2&-2&1&3\\0&0&1&1&-1\\0&0&0&1&0\\0&0&0&10&0\end{pmatrix}$$

$$\xrightarrow[\substack{R_2\to R_2-R_3\\R_1\to R_1-R_3}]{R_4\to R_4-10R_3}\begin{pmatrix}1&-2&-2&0&3\\0&0&1&0&-1\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix}\xrightarrow{R_1\to R_1+2R_2}\begin{pmatrix}1&-2&0&0&1\\0&0&1&0&-1\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix}.$$

(a) $\{(1,-2,0,0,1),(0,0,1,0,-1),(0,0,0,1,0)\}$ constitutes a basis for $Row(A)$.

(b) $\{(1,3,1,-2)^T,(-2,-1,5,-3)^T,(1,2,8,1)^T\}$ constitutes a basis for $Col(A)$.

(c) $\{(-1,0,1,0,1)^T,(2,1,0,0,0)^T\}$ constitutes a basis for $Null(A)$.

8. (8 marks) Let P_3 be the set of polynomials of degree less than 3 with real coefficients.

(a) Determine whether the following sets are linearly independent in P_3.

(i) $1-x$, $x-x^2$, $1-x^2$

(ii) $1+2x$, $x+2x^2$, $2+x^2$

(b) Let $p_1(x),p_2(x),p_3(x)\in P_3$ and define $\mathbf{v}_k=(p_k(-1),p_k(0),p_k(1))\in\mathbb{R}^3$, for $k=1,2,3$. Prove that if $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ are linearly independent, then $p_1(x),p_2(x),p_3(x)$ are linearly independent.

(a)**Soution:** (i)

$$c_1(1-x)+c_2(x-x^2)+c_3(1-x^2)=0,$$

$$(c_1+c_3)+(-c_1+c_2)x+(-c_2-c_3)x^2$$

The equation is equivalent to the following linear system

$$\begin{cases} c_1 + c_3 = 0\\ -c_1 + c_2 = 0\\ -c_2 - c_3 = 0\end{cases}$$

The augmented coefficient matrix of this system $\left(\begin{array}{ccc|c}1&0&1&0\\-1&1&0&0\\0&-1&-1&0\end{array}\right)$ reduces to the echelon form $\left(\begin{array}{ccc|c}1&0&1&0\\0&1&1&0\\0&0&0&0\end{array}\right)$, which implies that $c_1=1,c_2=1,c_3=-1$.

Therefore $1-x$, $x-x^2$, $1-x^2$ are linearly dependent in P_3.

(ii)

$$c_1(1+2x)+c_2(x+2x^2)+c_3(2+x^2)=0,$$

$$(c_1 + 2c_3) + (2c_1 + c_2)x + (2c_2 + c_3)x^2$$

The equation is equivalent to the following linear system

$$\begin{cases} c_1 & & + & 2c_3 & = & 0 \\ 2c_1 & + & c_2 & & = & 0 \\ & & 2c_2 & + & c_3 & = & 0 \end{cases}$$

The augmented coefficient matrix of this system $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$ reduces to the echelon form $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$, which implies that $c_1 = 0, c_2 = 0, c_3 = 0$.

Therefore $1 + 2x$, $x + 2x^2$, $2 + x^2$ are linearly independent in P_3.

(b)

Proof. Suppose that $c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0$.
Let $x = -1, 0, 1$, then we have

$$c_1p_1(-1) + c_2p_2(-1) + c_3p_3(-1) = 0,$$

$$c_1p_1(0) + c_2p_2(0) + c_3p_3(0) = 0,$$

$$c_1p_1(1) + c_2p_2(1) + c_3p_3(1) = 0,$$

which implies that $c_1v_1 + c_2v_2 + c_3v_3 = 0$.
Since v_1, v_2, v_3 are linearly independent, we have $c_1 = c_2 = c_3 = 0$.
Therefore, $p_1(x), p_2(x), p_3(x) \in P_3$ are linearly independent.